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Gel'fand-Zakharevich Systems and Algebraic Integrability: the Volterra Lattice Revisited

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Abstract

In this paper we will discuss some features of the bi-Hamiltonian method for solving the Hamilton-Jacobi (H-J) equations by Separation of Variables, and make contact with the theory of Algebraic Complete Integrability and, specifically, with the Veselov–Novikov notion of *algebro-geometric (AG) Poisson brackets*. The bi-Hamiltonian method for separating the Hamilton-Jacobi equations is based on the notion of pencil of Poisson brackets and on the Gel'fand-Zakharevich (GZ) approach to integrable systems. We will herewith show how, quite naturally, GZ systems may give rise to AG Poisson brackets, together with specific recipes to solve the H-J equations. We will then show how this setting works by framing results by Veselov and Penskoï about the algebraic integrability of the Volterra lattice within the bi-Hamiltonian setting for Separation of Variables

1 Introduction

The Hamilton-Jacobi equations, and the problem of their separability, are one of the many fields in mathematics in which the influence and heritage of Carl Gustav Jacob Jacobi is still alive. Such a problem, which can be considered one of the fundamental problems of Theoretical Mechanics, is rooted in the foundational works of Jacobi, Stäckel, Levi-Civita and others. It has recently

received a strong renewed interest thanks to its applications to the theory of integrable PDEs of KdV type (namely, the theory of finite-gap integration) and to the theory of quantum integrable systems (see, e.g., [6, 20]).

The constructive definition of separability originally due to Jacobi is the following.

Let us consider an *integrable* Hamiltonian H on a $2n$-dimensional phase space, that is, let us suppose that, along with $H = H_1$ we have further $n-1$ mutually commuting integrals of the motion $H_2, \dots H_n$, with $dH_1 \wedge \dots \wedge dH_n \neq 0$.

Definition 1.1 *An integrable system $(H_1, \dots, H_n)$ is separable in the canonical coordinates $(\mathbf{p}, \mathbf{q})$ if there exist n independent relations*

$$\Phi_i(q_i, p_i; H_1, \dots, H_n) = 0, \quad i = 1, \dots n\,, \tag{1.1}$$

connecting single pairs (q_i, p_i) *of coordinates with the n Hamiltonians H_j.*

The link of this definition with the theory of those integrable systems that admit a Lax representation with spectral parameter

$$\dot{L}(\lambda) = [L(\lambda), P(\lambda)],$$

such as those associated with classical limits of quantum spin systems and/or those coming from suitable reductions/restrictions of KdV-like evolutionary PDEs, is self evident. Indeed, the Lax representation of a system provides us with a natural candidate for the separation relations: the characteristic polynomial of $L(\lambda)$, also known as the *spectral curve* associated with $L(\lambda)$. However, the possibility of successfully applying the Lax method relies on three non-algorithmic steps to be solved:

- *To find* the Lax representation of a dynamical system;
- To prove that the spectral invariants of $L(\lambda)$ are mutually in involution (i.e., to prove that the Lax representation is compatible with a classical r-matrix structure);
- To give canonical coordinates as the coordinates of n points lying on the spectral curve, i.e., to actually implement what is sometimes called *Sklyanin's magic recipe*.

The setting devised by Veselov and Novikov [22] to characterize algebraic integrability requires that the phase space $\mathcal{M}$ of a Hamiltonian system fulfill some properties. They can be, quite roughly, summarized as follows:

a) $\mathcal{M}$ has the fibered structure

$$\mathcal{M} \xrightarrow{S^k\Gamma} \mathcal{B}, \tag{1.2}$$

where the base $\mathcal{B}$ is a n-dimensional manifold whose points determine an algebraic curve $\Gamma(b)$, and the fiber is the k–th symmetric product of that curve. In more details, one requires that $\Gamma(b)$ be given as an m–sheeted covering $\Gamma(b) \xrightarrow{m} \mathbb{C}$ of the complex λ-plane, and that points of $\mathcal{M}$ can be parameterized via the curve $\Gamma(b)$, and a set of k points on it, that is, the coordinates $\lambda_1, \ldots, \lambda_k$ of the projection on the λ-plane of a set of points on it, as well as discrete parameters ϵ_i that specify on which sheet of the covering the points live.

b) An Abelian differential $Q(\Gamma)$ on Γ (or possibly on a covering of Γ), smoothly depending on the points $b \in \mathcal{B}$, is defined. It is furthermore required that, if $Q(\Gamma)$ is given by

$$Q(\Gamma) = Q(b; \lambda) d\lambda \tag{1.3}$$

according to the representation of Γ as a covering of the λ-plane, the closed two-form

$$\omega_Q = \sum_{i=1}^{k} dQ(b; \lambda_i) \wedge d\lambda_i \tag{1.4}$$

give rise to a Poisson bracket, conveniently called *algebro-geometric* Poisson bracket, with λ_i and $\mu_i = Q(b; \lambda_i)$ playing the role of Darboux coordinates on the symplectic leaves of this bracket.

In such a case, it was proven in [22] that functions that depend only on the curve Γ – i.e., on the points of $\mathcal{B}$ – are in involution with respect to the Poisson bracket defined by (1.4), and these geometric data explicitly define action-angle variables for these Hamiltonian flows. In that fundamental paper it has also been shown that a number of integrable systems, of classical (i.e., mechanical) type as well as obtained by suitable reductions of soliton equations, can be framed within such a scheme. In particular, the paper [23] shows how the Volterra lattice fits in it. Finally, it is worth mentioning that Sklyanin's method [20] of the poles of the Baker-Akhiezer function, originally introduced in the study of Hamiltonian systems as a byproduct of "Quantum Integrability", can be seen as a particularly efficient scheme of implementing the Veselov–Novikov axiomatic picture.

More recently, a bi-Hamiltonian approach to Separation of Variables (SoV), has been exposed in the literature (see, e.g., [3, 9, 8, 18, 11, 2]). Such a scheme can be seen as a kind of bridge between the classical and the modern points of view, putting an emphasis on the geometrical structures of the Hamiltonian theory. In this framework it is possible to formulate intrinsic conditions on the integrable system $(H_1, \ldots, H_n)$ to *a priori* ensure separability in a set of canonical coordinates. It requires the existence, on the phase space (M, ω), of a *second* Hamiltonian structure, compatible with the one defined by ω. Namely, the bi-Hamiltonian structure on M allows, as it has been shown in a number of examples,

1. To encompass the definition of a special set of coordinates, to be called *Darboux–Nijenhuis (DN)* coordinates, within a well defined geometrical object.

2. To formulate intrinsic (i.e., tensorial) conditions for the separability of a Hamiltonian integrable system, in the DN coordinates associated with the bi-Hamiltonian structure.

3. To give recipes to characterize, find and handle sets of DN coordinates.

In particular, in [11] a detailed discussion of the bi-Hamiltonian scheme for SoV in the case of Gel'fand-Zakharevich type [13] systems was presented. It was also pointed out how the separation relations of such systems were, under genericity assumptions, of *degenerate* type, that is, the functional form of the separation relations (1.1) is the same for all pairs of separation coordinates (λ_i, μ_i), which essentially means that these coordinates are the coordinates of different points on the same algebraic curve.

In this paper we want to elaborate further on this issue, and, in particular, establish a connection between the bi-Hamiltonian scheme and the VN setting. This will be done in the first part of the paper, and, namely, in Proposition 3.3, which shows that (under suitable assumptions) DN coordinates associated with Gel'fand-Zakharevich systems can be seen as algebro-geometric canonical coordinates in the VN sense. In the second part of the paper we will apply our scheme in revisiting the algebro-geometrical integrability [19, 23] of the well-known Volterra lattice.

More in details, the paper is organized as follows. In Section 2 we will briefly review the bi-Hamiltonian set–up for SoV. In Section 3, we will recall some results exposed in [11], and discuss the relations of the bi-Hamiltonian approach with the VN scheme. In Section 4 we will collect a few results

concerning the algebro-geometric scheme of integrating the Volterra lattice. Finally, in Section 5 we will show how the bi-Hamiltonian picture of Section 3 can be successfully applied to the lattice, with no significant differences between the cases with odd (resp., even) number of sites.

2 Bi-Hamiltonian geometry and Separation of Variables

The basic geometrical notion underlying the bi-Hamiltonian scheme for separation variables is that of "semisimple ωN manifold". An ωN manifold is a symplectic manifold (M, ω) endowed with a second (possibly degenerate) Poisson tensor P_1 which is compatible with the Poisson tensor P_0 associated with the symplectic form ω. This means that $P_\lambda = P_1 - \lambda P_0$ is a Poisson tensor for all $\lambda \in \mathbb{R}$. In this case P_λ is called the Poisson pencil and (M, P_0, P_1) is a bi-Hamiltonian manifold. It can be shown (see, e.g., [15]) that the (1,1) tensor field $N = P_1 \circ {P_0}^{-1}$ has the property

$$[NX, NY] = N\big([NX, Y] + [X, NY] - N([X, Y])\big), \tag{2.1}$$

for all vector fields X, Y on M, that is, its Nijenhuis torsion vanishes. The tensor field N is called *Nijenhuis tensor* or *recursion operator* of the ωN manifold. It turns out that the characteristic polynomial of N is the square of a polynomial $\Delta(\lambda)$; the ωN manifold M is called semisimple provided that the roots of $\Delta(\lambda)$ be (generically) simple.

A special class of coordinates, to be called Nijenhuis coordinates, are provided by the spectral analysis of the adjoint recursion operator $N^* = P_0^{-1} \circ P_1$. Indeed, one has the following results (see, e.g., [14, 16, 15]):

1. The eigenspace Λ_i corresponding to any root λ_i is an integrable two-dimensional codistribution, that is, one can find n pairs of functions f_i, g_i (to be called Nijenhuis coordinates) satisfying

$$N^* df_i = \lambda_i df_i, \quad N^* dg_i = \lambda_i dg_i; \tag{2.2}$$

2. The eigenspaces Λ_i and Λ_j are orthogonal with respect to the Poisson brackets induced both by P_0 and P_1. This crucial property can be very simply proven. Indeed, let f and g be such that their differentials

belong respectively to Λ_i and to Λ_j, with $i \neq j$. Then one has, on the one hand:

$$\{f, g\}_1 = \langle df, P_1 dg\rangle = \langle N^* df, P_0 dg\rangle = \lambda_i \{f, g\}_0.$$

Switching the role of f and g one sees that $\{f, g\}_1 = \lambda_j \{f, g\}_0$, whence the assertion.

3. Since (as it is easy to prove) the Poisson bracket $\{f_i, g_i\}_0$ with respect to P_0 (and also the one with respect to P_1) of functions that satisfy (2.2) still satisfies $N^* d\{f_i, g_i\}_0 = \lambda_i \{f_i, g_i\}_0$, it is possible to parameterize Λ_i with a set of coordinates x_i, y_i, called Darboux-Nijenhuis coordinates, that are Nijenhuis coordinates and are *canonical* for ω (whence the addition of "Darboux" in their denomination).

4. It is important to notice that the Nijenhuis tensor N of an ωN manifold defines, at each point $m \in M$, a linear operator $N_m : T_m M \to T_m M$. As such, its eigenvalues (that are point-wise the roots of $\det(N_m - \lambda)$) may depend on the point m. In this case we will call these roots, *nonconstant* roots of $\Delta(\lambda)$. If $\bar{\lambda}$ is a nonconstant root of $\Delta(\lambda)$, then it satisfies the characteristic equation[1] (2.2),

$$N^* d\bar{\lambda} = \bar{\lambda} d\bar{\lambda}$$

The following proposition has been proven in [11].

Proposition 2.1 *Suppose that (M, ω, P_1) is a semisimple ωN manifold of dimension $2n$. Let $x_1, \ldots, x_n, y_1, \ldots x_n$ be Darboux-Nijenhuis coordinates on M and let $F_1, \ldots, F_n$ be functionally independent Hamiltonians, that are in involution with respect to the Poisson brackets induced by P_0 and P_1. Then the Hamilton-Jacobi equations associated with any of the Hamiltonians F_i can be solved by additive separation of variables in the Darboux-Nijenhuis coordinates $(x_i, y_i)_{i=1,\ldots,n}$.*

The foliation given by the functions F_j will be called a *bi-Lagrangian foliation*. Such foliations provide a geometrical description of separable systems, exactly like Lagrangian foliations describe integrable systems.

To elaborate further on the geometric structure of ωN manifold, it is convenient to suppose that the eigenvalues of the Nijenhuis tensor N be

[1] Actually, the constant eigenvalues trivially satisfy the same equation.

functionally independent. A straightforward observation [14, 11] shows that one can compactly write the characteristic equation $N^* d\lambda_i = \lambda_i d\lambda_i$ in terms of the minimal polynomial

$$\Delta_N(\lambda) = \prod_{i=1}^{\frac{1}{2}\dim(M)} (\lambda - \lambda_i)$$

as the polynomial relation

$$N^* d\Delta_N(\lambda) = \lambda d\Delta_N(\lambda). \tag{2.3}$$

Actually, relations of this kind are very important for our purposes. Indeed, in [11] we proved the following proposition:

Proposition 2.2 *Let $\Phi(\lambda)$ be a smooth function defined on the ωN manifold M, depending on an additional parameter λ. Suppose that there exists a one-form α_Φ such that*

$$N^* d\Phi(\lambda) = \lambda d\Phi(\lambda) + \Delta_N(\lambda)\alpha_\Phi \, . \tag{2.4}$$

where Δ_N is the minimal polynomial of N. Then:
a) the n functions Φ_i obtained evaluating the "generating" function $\Phi(\lambda)$ for $\lambda = \lambda_i, i = 1, \ldots, n$ are Nijenhuis functions, that is, they satisfy $N^* d\Phi_i = \lambda_i d\Phi_i$.
b) If $\Phi(\lambda)$ satisfies (2.4) and $Y_l = -P_0 dp_l$ are the vector fields associated via P_0 to the coefficients of the minimal polynomial of the Nijenhuis tensor, then all functions

$$\Phi_l(\lambda) = Lie_{Y_l}(\Phi(\lambda)))$$

satisfy (2.4) as well.
c) In particular, if $\Phi(\lambda)$ satisfies, along with (2.4), the relation

$$Lie_{Y_1}(\Phi(\lambda)) = 1 \qquad \mod \Delta_N(\lambda), \tag{2.5}$$

then the functions $(\lambda_i, \mu_i := \Phi(\lambda_i))$ provide a set of Darboux-Nijenhuis coordinates on M.

Definition 2.3 *We will call a generating function $\Phi(\lambda)$ satisfying equation (2.4) a* Nijenhuis functions generator.

3 Gel'fand-Zakharevich systems

In many of the models considered in the so-called "Modern Theory of Integrable Systems", that is, finite-dimensional Hamiltonian systems obtained as reductions of integrable PDEs, and/or classical analogs of quantum spin systems, two instances occur:

- The phase space M of the system is not the cotangent bundle to a smooth manifold Q.
- M is endowed with a pair of compatible Poisson brackets, but none of them is nondegenerate.

This is, for instance, the case of the Volterra lattice. This geometrical instance has been formalized in a series of papers by Gel'fand and Zakharevich. In particular, under some technical assumptions, one of which is that the corank of P_0 equals that of P_1, it is possible to construct $N = \mathrm{corank}(P_0)$ Lenard–Magri sequences that start with a Casimir function of P_0 and end with a Casimir function of P_1. The Hamiltonians of these sequences can be conveniently collected in polynomials $H^{(a)}(\lambda)$, $a = 1, \dots, N$, in the variable λ, satisfying

$$(P_1 - \lambda P_0) dH^{(a)}(\lambda) = 0, \tag{3.1}$$

called *polynomial Casimirs* of the *pencil*. It is nowadays customary to denote such a geometrical instance of bi-Hamiltonian manifolds (M, P_0, P_1) with the name of *Gel'fand–Zakharevich (GZ)* manifold.

The collection of the degrees of the polynomial Casimirs is a numeric invariant of the bi-Hamiltonian manifold. Notice in particular that Casimirs of degree 0 are nothing but *common* Casimir functions of the two Poisson tensors. We will, in the sequel, call these common Casimirs *trivial Casimirs*, and refer to the others (namely, those originating a non void Lenard–Magri sequence) as *nontrivial ones*.

In the case of bi-Hamiltonian systems associated with (coefficients of) polynomial Casimirs, one can try to use the bi-Hamiltonian scheme for SoV reducing the systems to a suitable ωN manifold. More precisely, one can consider a symplectic leaf $S \subset M$ of P_0 and a suitable deformation of the Poisson pencil, discussed in detail in [10, 11, 5, 17].

First one fixes a maximal set $C_1, \dots, C_k$ of independent nontrivial Casimirs of P_0, and finds k (independent) vector fields $Z_1, \dots, Z_k$ such that

$$\mathrm{Lie}_{Z_a}(C_b) = \delta_{ab}, \qquad \mathrm{Lie}_{Z_a} K_\alpha = 0,$$

where K_α are the common Casimirs[2]. We will hereinafter refer to the distribution $\mathcal{Z}$ generated by the k vector fields Z_i as the *transversal distribution*.

Then one considers the vector fields $X_a = P_1 dC_a$, for $a = 1, \ldots, k$, and the "deformed" tensor

$$\widetilde{P_1} = P_1 - \sum_{a=1}^{k} X_a \wedge Z_a, \tag{3.2}$$

that restricts to S. If the algebra of functions vanishing along $Z_1, \ldots, Z_k$ is a Poisson algebra for the pencil $P_1 - \lambda P_0$, it turns out [10] that the deformation $\widetilde{P_1}$ defines on S a Poisson tensor $\widetilde{P_1}|S$ compatible with the restriction $P_0|S$ of P_0 to its symplectic leaves. So S is endowed with the structure of a ωN manifold. By the definition of $\widetilde{P_1}$, the restrictions of the coefficients $H_l^{(a)}$ of $H^{(a)}(\lambda)$ to S will be separable Hamiltonians in the Darboux-Nijenhuis coordinates defined on S (provided S is semisimple).

Remark. To summarize, the ideas underlying such a reduction procedure are the following: the Gel'fand-Zakharevich scheme provides – under some technical assumptions – a way for defining, via Magri-Lenard sequences, a distinguished integrable distribution $\mathcal{A}$ on a bi-Hamiltonian manifold (M, P_0, P_1). It is called the *axis* of M, and is generated by the Hamiltonian vector fields associated with the coefficients $H_i^{(a)}$ of the Casimir polynomials, so that the leaves $\mathcal{F}$ of $\mathcal{A}$ are defined by the requirement that these coefficients be constant along $\mathcal{A}$.

If we fix our attention on one of the elements of the Poisson pencil, say, P_0, and consider its symplectic foliation $\mathcal{S}$, we have that $\mathcal{F} \cap \mathcal{S}$ defines, on the generic symplectic leaf S of $\mathcal{S}$, a Liouville integrable system.

However, since in general the symplectic foliation associated with the other Poisson tensor P_1 does not coincide with $\mathcal{S}$, S does not come equipped with a natural bi-Hamiltonian structure. Finding the distribution $\mathcal{Z}$ with the properties outlined above amounts to finding a deformation $\widetilde{P_1}$ of P_1 that

a) endows the symplectic manifold S with a compatible second Poisson tensor, and hence with the structure of a ωN manifold.

b) Preserves the commutativity of the Hamiltonians $H_i^{(a)}$, that is, provides $\mathcal{F} \cap \mathcal{S}$ with the structure of a bi-Lagrangian foliation.

[2]In the papers referred to above, the distribution $\mathcal{Z}$ spanned by $Z_1, \ldots, Z_k$ was required to be satisfy the stronger condition $T_pM = T_pS \oplus \mathcal{Z}_p$ for all $p \in S$. Actually, as it should be clear form [11], common Casimirs do not enter the reduction procedure. A similar instance with common Casimirs has been considered in [12].

Although to find the transversal distribution $\mathcal{Z}$ is a non algorithmic procedure, we notice that this is a quite efficient way of providing the symplectic leaves S of P_0 with a compatible Nijenhuis tensor (a problem which, in principle, requires the solution of a system of nonlinear partial differential equations).

Definition 3.1 *We say that a GZ manifold (M, P_0, P_1), endowed with a transversal distribution $\mathcal{Z}$ satisfying the above mentioned assumptions, admits an affine structure if it is possible to choose a complete set of nontrivial Casimirs of P_0, and a corresponding basis of* normalized flat generators $\{Z_b\}_{b=1,\dots,k}$ *in $\mathcal{Z}$ such that, for every Casimir of the Poisson pencil $H^a(\lambda)$ and every b, c, one has the vanishing of the second Lie derivative of the Casimir polynomials:*

$$Lie_{Z_b} Lie_{Z_c}(H^a(\lambda)) = 0. \tag{3.3}$$

The above definition might seem somewhat *ad hoc*. Its relevance can be summarized in the following points (whose proof can be, once more, found in [11]) that hold in the case of affine structures.

- The nonconstant roots λ_i of the determinant of the matrix whose entries are
$$\mathcal{G}_{ab} = \mathrm{Lie}_{Z_b}(H^a(\lambda))$$
satisfy
$$N^* d\lambda_i = \lambda_i d\lambda_i,$$
namely, they are roots of the minimal polynomial of the Nijenhuis tensor induced on (any of) the symplectic leaves of P_0 by the pair $(P_0, \widetilde{P}_1)$. In particular, if there is only one non-trivial Casimir polynomial $H(\lambda)$, and Z is a corresponding normalized flat generator, the nonconstant roots of the polynomial $\mathrm{Lie}_Z(H(\lambda))$ are "nonconstant" eigenvalues of the Nijenhuis tensor N.

- The separation relation satisfied by the non-trivial Hamiltonian functions and the Darboux-Nijenhuis coordinates are *linear* in the Hamiltonians, that is, are of (generalized) Stäckel type.

The following Lemma, whose proof is a simple application of some notions of Poisson geometry, will be frequently used in the sequel. It provides a link between the properties of functions on the GZ manifold M, which depend

polynomially on the parameter λ of the Poisson pencil $P_1 - \lambda P_0$ defined on M, and the properties of the evaluation of such functions in $\lambda = \lambda_i$ w.r.t. the induced Nijenhuis structure on the symplectic leaves. In plain words, it allows us to work on the GZ manifold M, without having to actually perform the reduction procedure.

We still suppose that (M, P_0, P_1) is a GZ manifold, with k non-trivial Lenard-Magri sequences. We suppose that $Z_1, \ldots, Z_k$ are normalized transversal generators for the distribution $\mathcal{Z}$ we considered above. We recall that, in this situation, the symplectic leaves of P_0 are ωN manifolds, with induced Nijenhuis tensor $N = P_0^{-1}\widetilde{P_1}$..

Lemma 3.2 *Let F_λ be a function on M, invariant along the fields Z_i, that depends holomorphically (say, polynomially) on an additional parameter λ; its restriction f_λ to a symplectic leaf S of P_0 satisfies the "eigenvector" equation*

$$N^* df_\lambda \big|_{\lambda=\lambda_i} = \lambda_i df_\lambda \big|_{\lambda=\lambda_i}$$

for all eigenvalues λ_i if and only if the following equality holds, parametrically in λ, on the GZ manifold M:

$$\{G, F_\lambda\}_{P_1} - \sum_{a=1}^{k} Lie_{Z_a}(G)\{K_1^{(a)}, F_\lambda\}_{P_0} = \lambda\{G, F\}_{P_0}, \tag{3.4}$$

for any $G \in C^\infty(M)$, where the $K_1^{(a)}$ satisfy $P_1 dC_a = P_0 dK_1^{(a)}$, $a = 1, \ldots, k$. Otherwise stated, we have to require that

$$\widetilde{P_1} dF_\lambda = \lambda P_0 dF_\lambda.$$

where $\widetilde{P_1}$ is defined in (3.2)

A direct generalization of the above proposition shows that spectral curves might be a source for finding Nijenhuis functions generators.

Proposition 3.3 *Let us consider a generating function $\Gamma(\lambda, \mu)$ of Casimirs of a Poisson pencil P_λ, and let us suppose that $\Gamma(\lambda, \mu) = 0$ defines a smooth algebraic curve. Let S be a generic symplectic leaf of P_0 and let N be the Nijenhuis tensor associated — according to the scheme outlined above — with P_λ and a suitable transversal distribution $\mathcal{Z}$. Suppose that f is a $\mathcal{Z}$-invariant root of the minimal polynomial of N, i.e.,*

$$N^* df = f df, \quad \text{and } Z_i(f) = 0,\ i = 1, \ldots, k, \tag{3.5}$$

and suppose that $\Gamma(\mu, f) = 0$ *defines generic point(s) of the affine curve* $\Gamma(\lambda, \mu) = 0$. *Then, any solution* g *of the equation* $\Gamma(g, f) = 0$ *which is invariant as well under* $\mathcal{Z}$ *satisfies* $N^* dg = f dg$.

Proof: We first notice the following. Let us consider a bivariate polynomial $F(\lambda, \mu) = \sum_{i,j} f_{(i,j)} \lambda^i \mu^j$, with coefficients $f_{(i,j)}$ that are functions defined on a manifold M, and two more distinguished functions on M, say f and g. If we define $\mathcal{F} := F(f, g)$, then:

$$d\mathcal{F} = dF(\lambda, \mu)\Big|_{\substack{\lambda=f \\ \mu=g}} + \frac{\partial F}{\partial \lambda}\Big|_{\substack{\lambda=f \\ \mu=g}} df + \frac{\partial F}{\partial \mu}\Big|_{\substack{\lambda=f \\ \mu=g}} dg. \tag{3.6}$$

We consider the equation of the "spectral curve", $\Gamma(\mu, \lambda) = 0$; so we get for the zeroes of the function $\mathcal{R} = \Gamma(f, g)$,

$$0 = d\Gamma(\lambda, \mu)\Big|_{\substack{\lambda=f \\ \mu=g}} + \frac{\partial \Gamma}{\partial \lambda}\Big|_{\substack{\lambda=f \\ \mu=g}} df + \frac{\partial \Gamma}{\partial \mu}\Big|_{\substack{\lambda=f \\ \mu=g}} dg. \tag{3.7}$$

Let us suppose, for simplicity, that $\Gamma(\lambda, \mu) = \Gamma_0(\lambda\mu) + \sum_{i=1}^k \mu^{n_i} H^i(\lambda)$, where $\Gamma_0(\mu, \lambda)$ is a constant polynomial (possibly depending on the common Casimirs), and n_i ares suitable integers. Considering the action of the k transversal vector fields Z_i, we get the k equations:

$$\mathrm{Lie}_{Z_i}(g) \frac{\partial \Gamma}{\partial \mu}\Big|_{\substack{\lambda=f \\ \mu=g}} + \mathrm{Lie}_{Z_i}(f) \frac{\partial \Gamma}{\partial \lambda}\Big|_{\substack{\lambda=f \\ \mu=g}} + \sum_{j=1}^k \mu^{n_j} \mathrm{Lie}_{Z_i}(H^j(\lambda))\Big|_{\substack{\lambda=f \\ \mu=g}} = 0. \tag{3.8}$$

Applying $\widetilde{P}_1 - f P_0$ to eq. (3.7) we get, using Lemma 3.2 and taking into account that $(P_1 - \lambda P_0) d\Gamma(\lambda, \mu) = 0$,

$$\frac{\partial \Gamma}{\partial \mu}\Big|_{\substack{\lambda=f \\ \mu=g}} (N^* - f) dg + \frac{\partial \Gamma}{\partial \lambda}\Big|_{\substack{\lambda=f \\ \mu=g}} (N^* - f) df - \sum_{i=1}^k \sum_{j=1}^k \left(g^{n_i} \mathrm{Lie}_{Z_j} H^i(\lambda) \right)\Big|_{\lambda=f} dH_1^j = 0. \tag{3.9}$$

Plugging into equations (3.8,3.9) the hypotheses on f we arrive at the system

$$\begin{cases} \Gamma(f, g) = 0 \\ \mathrm{Lie}_{Z_i}(g) \dfrac{\partial \Gamma}{\partial \mu}\Big|_{\substack{\lambda=f \\ \mu=g}} - \displaystyle\sum_{j=1}^n g^{n_j} \mathrm{Lie}_{Z_i}(H^j(\lambda))\big|_{\lambda=f} = 0, \quad i = 1, \ldots, k. \\ \dfrac{\partial \Gamma}{\partial \mu}\Big|_{\substack{\lambda=f \\ \mu=g}} (N^* - f) dg - \displaystyle\sum_{i=1}^n \sum_{j=1}^n g^{n_i} \mathrm{Lie}_{Z_j} H^i(\lambda)\big|_{\lambda=f} dH_1^j = 0. \end{cases} \tag{3.10}$$

The thesis follows noticing that if g is invariant under $\mathcal{Z}$ then this system reduces to

$$\begin{cases} \Gamma(f,g)=0 \\ \dfrac{\partial\Gamma}{\partial\mu}\Big|_{\substack{\lambda=f\\ \mu=g}}(N^*-f)dg=0, \end{cases} \tag{3.11}$$

and taking into account that, for $\Gamma(\lambda,\mu)$ smooth, the solution of system

$$\begin{cases} \Gamma(f,g)=0 \\ \dfrac{\partial\Gamma}{\partial\mu}\Big|_{\substack{\lambda=f\\ \mu=g}}=0 \end{cases}$$

are the (fixed) ramification points of $\Gamma(\lambda,\mu)=0$.

□

This proposition provides us with the desired link between the bi-Hamiltonian approach and the VN axiomatic picture of AG brackets. Indeed, it can be restated as follows: suppose we can find (by means of a Lax representation, or by other means) a generating function for the Casimirs polynomials of an affine GZ pencil in the form of a bivariate polynomial $\Gamma(\lambda,\mu)$. Then, suppose that the coordinates of the points on the curve $\Gamma(\lambda,\mu)=0$ (whose λ-projections give the roots of the minimal polynomial of the Nijenhuis tensor N induced on symplectic leaves of P_0) satisfy the invariance condition specified in the above proposition. Then they are Nijenhuis coordinates, and so their Poisson brackets are given by

$$\{\lambda_i,\mu_j\}_0=\delta_{ij}\varphi_i(\lambda_i,\mu_i),\quad \{\lambda_i,\mu_j\}_1=\delta_{ij}\lambda_i\varphi_i(\lambda_i,\mu_i),$$

Under the further assumption of irreducibility of the minimal polynomial of N, one sees that the unknown functions φ_i cannot explicitly depend on the index i. Thus the formal integral

$$Q(\gamma,\lambda)d\lambda=\left(\int^{\mu}\frac{d\nu}{\varphi(\lambda,\nu)}\right)d\lambda \tag{3.12}$$

will give the VN meromorphic differential defining Algebro-geometrical Poisson brackets corresponding to P_0 (as well as $Q'=\dfrac{Q}{\lambda}$ gives those corresponding to P_1).

4 The Volterra Lattice

The Volterra lattice equations are the following set

$$\dot{c}_i = c_i(c_{i+1} - c_{i-1}), \tag{4.1}$$

which we consider to be defined on a periodic lattice $c_i > 0$, $c_{i+n} \equiv c_i$. They are generalization of the famous Volterra equations describing time evolution of competing species.

The phase space of the VL can be seen as the restriction to the submanifold of vanishing momenta of the periodic n-site Toda Lattice. It is well-known [7, 19, 23, 21, 4] that eq. (4.1) are isospectral deformation equations for the periodic difference second order operators of the form

$$(\mathcal{L}\psi)_k = a_{k+1}\psi_{k+1} + a_k\psi_{k-1}, \quad a_{n+i} = a_i,\ \psi_{n+i} = \lambda\psi_i. \tag{4.2}$$

where $a_k = \sqrt{c_k}$. In complete analogy with the Toda case, it admits a *dual* Lax representation[3] in terms of a 2×2 matrix $\mathcal{L}'$ being given by the (ordered) product of site matrices

$$\mathcal{L}' = \ell_n\ell_{n-1}\cdots\ell_1, \quad \ell_i(\mu) = \begin{pmatrix} \mu & c_i \\ -1 & 0 \end{pmatrix}.$$

The Volterra Lattice equations admit a bi-Hamiltonian formulation. Indeed, if one considers the quadratic Poisson tensor

$$P^{ij} = c_ic_j(\delta_{i+1,j} - \delta_{j+1,i}) \tag{4.3}$$

and the cubic one

$$Q^{ij} = c_ic_j(c_i + c_j)(\delta_{i+1,j} - \delta_{j+1,i}) + c_ic_{i+1}c_{i+2}\delta_{i+2,j} - c_ic_{i-1}c_{i-2}\delta_{i-2,j} \tag{4.4}$$

one notices that (4.1) can be written as

$$\dot{c}_i = \sum_j P^{ij}\frac{\partial h}{\partial c_j} = \sum_j Q^{ij}\frac{\partial k}{\partial c_j}, \tag{4.5}$$

where

$$h = \frac{1}{2}\log\prod_{i=1}^{n} c_i, \quad k = \sum_{i=1}^{n} c_i. \tag{4.6}$$

[3]This duality involves also, as in the case of the Toda Lattice, an exchange of the roles between the spectral parameter λ and the eigenvalue μ.

We can collect those remarkable results by Penskoï and Veselov-Penskoï [19, 23], that will be used in the sequel, as follows.

Let us set $\pi = (\prod_{i=1}^{n} c_i)^{1/2}$, and consider the (normalized) characteristic equation

$$\Gamma(\lambda, \mu) \equiv \frac{1}{\pi}\mathrm{Det}(\mu - \mathcal{L}(\lambda)) = 0. \tag{4.7}$$

Then it holds:

1. $\Gamma(\lambda, \mu) = \lambda + \frac{1}{\lambda} - H(\mu)$, where the polynomial $H(\mu)$ is expressed as

$$H(\mu) = \begin{cases} \sum_{i=0}^{k}(-1)^i \mu^{2k+1-2i}\dfrac{J_i}{\pi} = \\ \quad \mu \cdot (\sum_{i=0}^{k}(-1)^i \mu^{2(k-i)}\mathcal{H}_i), \quad \text{if } n = 2k+1 \\ \sum_{i=0}^{k+1}(-1)^i \mu^{2k+2-2i}\dfrac{J_i}{\pi} = \\ \quad \mu^2 \cdot (\sum_{i=0}^{k}(-1)^i \mu^{2(k-i)}\mathcal{H}_i) + \mathcal{H}_{k+1}, \quad \text{if } n = 2k+2 \end{cases} \tag{4.8}$$

   The functions J_i can be usefully characterized in the following way: J_i is the sum of all possible monomials $c_{l_1} c_{l_2} \cdots c_{l_i}$ of lenght i, where the indices l_p are all different and not congruent to 1 modulo n. In [23] subsets $\{l_1, \ldots, l_i\} \subset \{1, \ldots n\}$ satisfying this property are called *totally disconnected* and we will adopt this definition in the next subsection. Notice that, e.g., $J_0 = 1$, $J_1 = \sum_i c_i$. Furthermore, notice that for $n = 2k+2$ the last Hamiltonian $\mathcal{H}_{k+1}$ is given by

$$\mathcal{H}_{k+1} = A + \frac{1}{A}, \text{ with } A = \sqrt{\frac{c_2\, c_4 \cdots c_{2k+2}}{c_1\, c_4 \cdots c_{2k+1}}}. \tag{4.9}$$

2. (Theorem 3 of [19]). The functions $\mathcal{H}_i$ satisfy the Lenard-Magri recursion relations

$$Pd\mathcal{H}_0 = 0, \quad Pd\mathcal{H}_i = Qd\mathcal{H}_{i-1}, i = 1, \ldots, k, \quad Qd\mathcal{H}_k = 0. \tag{4.10}$$

   For $n = 2k+2$ the last Hamiltonian $\mathcal{H}_{k+1}$ is in the kernel of both P and Q.

3. Let $\{\zeta_1, \ldots, \zeta_k\}$ be a suitably chosen subset of k poles of a suitably normalized Baker-Akhiezer function Ψ associated with the Lax operator $\mathcal{L}$

of (4.2), and let λ_i be corresponding coordinates on the spectral curve Γ. Then the coordinates $\{\zeta_i, \rho_i = \frac{2\log|\lambda_i|}{\zeta_i}\}$ parametrize the symplectic leaves of P and satisfy

$$\{\zeta_i, \rho_j\}_P = \delta_{ij}, \quad \{\zeta_i, \rho_j\}_Q = \zeta_i^2 \delta_{ij}. \tag{4.11}$$

We remark that one can compactly restate the results of item 2) in the following form:

Proposition 4.1 *Let us define the Poisson pencil*

$$P_{\mu^2} = Q - \mu^2 P,$$

where P and Q are the quadratic and cubic Poisson tensors (4.3,4.4). Then the polynomial $H(\mu)$ defined by (4.8) is a Casimir polynomial of the pencil P_{μ^2}. Moreover, for $n = 2k+2$ the function $\mathcal{H}_{k+1}$ is a common Casimir of the two basic elements P and Q of the pencil. Since functional independence of the Hamiltonians $\mathcal{H}_i$ is self-evident, we see that, in the GZ terminology, the phase space M of the n-site Volterra lattice, equipped with the Poisson structures P and Q, is a GZ bi-Hamiltonian manifold. If $n = 2k+1$ or $n = 2k+2$, we have a single non-trivial Lenard-Magri sequence comprising k vector fields. If $n = 2k+2$ is even, the GZ manifold has a trivial Casimir $\mathcal{H}_{k+1}$.

5 Separation of Variables for the Volterra Lattice in the bi-Hamiltonian setting

In this section we will make the final contact between the known results we collected above, and explicitly show how the picture of [23] can be naturally framed within the bi-Hamiltonian theory of Separation of Variables for GZ systems described in Section 2

Our starting point are the GZ formulation of Penskoĭ's results, collected in Section 3, as well as the normalized spectral curve equation (4.8). Obviously enough, we will set, as natural parameter of the Poisson pencil, the quantity $\nu = \mu^2$.

We remark that both in the even and odd number of sites (or species) as there is only one non-trivial Lenard–Magri chain, we have to look for a single transversal vector field to deform the Poisson pencil $Q - \nu P$.

Let us consider

$$Z_0 = c_1 \frac{\partial}{\partial c_1} + c_n \frac{\partial}{\partial c_n}, \tag{5.1}$$

and define

$$X = Q d\mathcal{H}_0. \tag{5.2}$$

The vector field Z is the required ingredient for applying the bi-Hamiltonian setting to the Volterra lattice. This follows from the properties we list and prove below in a series of steps.

a) Z_0 is a symmetry for the quadratic Poisson tensor P,

$$\mathrm{Lie}_{Z_0} P = 0.$$

This follows noticing that the Jacobian of Z_0 with respect to the coordinates $(c_1, \ldots, c_0)$ is, in terms of the standard generators E_{ij} of $n \times n$ matrices, given by $E_{11} + E_{nn}$, and by the explicit form of P.

b) Still taking this property into account, one can easily verify that the action of Z_0 on Q is given by

$$\mathrm{Lie}_{Z_0} Q = Z_0 \wedge W_0, \tag{5.3}$$

where

$$W_0 = c_1 c_n \left(\frac{\partial}{\partial c_1} - \frac{\partial}{\partial c_n} \right) + c_1 c_2 \frac{\partial}{\partial c_2} - c_{n-1} c_n \frac{\partial}{\partial c_{n-1}}. \tag{5.4}$$

c)

$$\mathrm{Lie}_{Z_0}(\mathcal{H}_0) = -\mathcal{H}_0. \tag{5.5}$$

d) If $n = 2k + 2$, then

$$\mathrm{Lie}_{Z_0} \mathcal{H}_{k+1} = 0, \tag{5.6}$$

where $\mathcal{H}_{k+1}$ is given by (4.9). These last two properties can be easily verified by straightforward computations.

Hence we can state

Proposition 5.1 *Let $Z := -\dfrac{1}{\mathcal{H}_0} Z_0$ be the normalized symmetry of P. The two bivectors*

$$P, \quad \widetilde{Q} = Q - Z \wedge X \tag{5.7}$$

form a Poisson pencil that restricts to the (generic) symplectic leaf S of P. The functions $\mathcal{H}_i$ are in involution also with respect to the deformed Poisson bracket $\{\cdot,\cdot\}'$ associated with the bivector $\widetilde{Q}$; hence their restrictions $\widehat{\mathcal{H}}_i$ define a bi-Lagrangian foliation of S.

We are now left with the characterization of the Nijenhuis coordinates associated with the restriction of the pencil $\widetilde{Q} - \nu P$ to the symplectic leaf S.

Let us consider the vector field Z and the Hamiltonians $\mathcal{H}_i$, with $i = 1, \ldots, k$ if $n = 2k + 1$, and $i = 1, \ldots, k + 1$ if $n = 2k + 2$.

Lemma 5.2

$$Lie_Z(\mathcal{H}_i) = Lie_Z\left(\frac{J_i}{\pi}\right) = J_i\Big|_{c_1=c_n=0} \quad , \tag{5.8}$$

that is, the Lie derivative of $\mathcal{H}_i$ with respect to Z is nothing but the numerator of $\mathcal{H}_i$, evaluated at $c_1 = c_n = 0$.

Proof. The proof is a simple chain of computations. We report it here since this Lemma is crucial for the conclusion of the paper.

We remark that, by the definition of the functions J_i and that of Z_0, we have

$$\mathrm{Lie}_{Z_0}(J_i) = J_i\Big|_{c_2=\cdots=c_{n-1}=0} \quad , \tag{5.9}$$

along with (eq (5.5)) $\mathrm{Lie}_{Z_0}(\mathcal{H}_0) = -\mathcal{H}_0$. Thus

$$\mathrm{Lie}_Z(\mathcal{H}_j) = -\big(\frac{1}{\mathcal{H}_0}\mathrm{Lie}_{Z_0}(\mathcal{H}_0 J_i)\big) = -(J_i - J_i\Big|_{c_2=\cdots=c_{n-1}=0}) = J_i\Big|_{c_1=c_n=0}. \tag{5.10}$$

□

From the Lemma above, it immediately follows

Proposition 5.3 *The polynomial $Lie_Z(H(\mu))$ factors as*

$$Lie_Z(H(\lambda)) = \mu^a \Delta(\nu), \tag{5.11}$$

where $a = 1$ if n is odd, $a = 2$ if n is even, and Δ is a monic degree k polynomial in $\nu = \mu^2$ which, thanks to (5.10), is invariant along Z.

From the results recalled after Definition 3.1, we thus recover the eigenvalues of the Nijenhuis tensor on S as the roots ν_i of $\Delta(\nu)$. Since they clearly are functionally independent, we can choose, as first half of Darboux-Nijenhuis coordinates, their square roots:

$$\zeta_i = \sqrt{\nu_i}. \tag{5.12}$$

Also, if we consider the normalized spectral curve relation, we see that the solutions λ_i of the equation

$$\Gamma(\lambda, \zeta_i) = 0 \tag{5.13}$$

are invariant under Z as well, so that choosing one of the two solutions of this equations will provide a natural candidate for the remaining half of Nijenhuis coordinates.

To show that actually this is the case, and simultaneously define a set of Darboux-Nijenhuis coordinates (that is, normalized Nijenhuis coordinates), we can rely, once more, on the results of [23, 19], thanks to the following

Proposition 5.4 *Let us consider the Lax operator $\mathcal{L}$ of (4.2) and let ψ be a Baker Akhiezer vector, for $\mathcal{L}$, normalized with $\psi_1 = 1$. Then on the k points $P_i = (\lambda_i, \zeta_i)$ chosen according to the above recipe, ψ has a pole.*

Proof. The normalized BA function ψ has poles in the zeros of the $(1,1)$ element of the classical adjoint matrix

$$(\mu - \mathcal{L})^{\vee}. \tag{5.14}$$

This is the determinant of the $(n-1) \times (n-1)$ matrix

$$M = \mu \mathbf{1} - \sum_{a=1}^{n-2} \sqrt{c_{a+1}} (\mathsf{E}_{a,a+1} + \mathsf{E}_{a+1,a}), \tag{5.15}$$

whose determinant equals the $\Delta(\mu^2)$ if n is odd, and $\mu\Delta(\mu^2)$ if n is even.

□

This shows that the functions $(\zeta_i, \lambda_i)_{i=1,\dots,k}$ selected according the bi-Hamiltonian scheme herewith presented do indeed coincide with those found by Veselov and Penskoï via the method of poles of the BA function.

As a closing remark, we notice that a set of canonical coordinates for the Volterra lattice can be obtained via "purely bi-Hamiltonian methods" as

follows. As it has been remarked in Proposition 2.2, a possible path is to use the Hamiltonian vector fields associated with the coefficients p_i of the minimal polynomial of the Nijenhuis tensor $\Delta(\lambda)$ to deform the polynomial Casimir of the Poisson pencil. In this way we obtain new polynomials that satisfy the characteristic equation of a Nijenhuis functions generator (2.4), and hence we can use them to generate Darboux-Nijenhuis coordinates.

For the sake of concreteness we will stick to the case of an odd number of sites n.

We consider the minimal polynomial of the induced Nijenhuis tensor, given by (according to Proposition 5.3, with $n = 2k + 1$),

$$\Delta(\nu) = \Big(\frac{1}{\mu}\mathrm{Lie}_Z(H(\mu))\Big)\Big|_{\mu^2=\nu} = \nu^k - p_1\nu^{k-1} - \cdots p_k, \text{ where } p_j = -\mathrm{Lie}_Z(\mathcal{H}_j). \tag{5.16}$$

Thanks to the explicit characterizations of the Hamiltonians $\mathcal{H}_i$ and of the vector field Z, and taking Lemma 5.2 into account, it is not difficult to ascertain that

$$p_k = \prod_{i=1}^{k} c_{2i}. \tag{5.17}$$

Keeping into account the explicit form of the quadratic Poisson tensor (4.3), the Hamiltonian vector field associated with $\log p_k$ is given by the very simple expression

$$Y = P\, d\log p_k = c_1\frac{\partial}{\partial c_1} - c_n\frac{\partial}{\partial c_n} \tag{5.18}$$

If we define as "first half" of the Darboux-Nijenhuis coordinates the *logarithms* of the eigenvalues of the Nijenhuis tensor, rather than their square roots, that is, if we consider

$$\phi_i = \log(\nu_i),$$

we clearly have that, in terms of the canonical coordinates ψ_i conjugated to the ϕ_i we are seeking, it holds

$$Y = P\sum_i d\phi_i = \sum_i \frac{\partial}{\partial\psi_i}, \tag{5.19}$$

so that according to the recipe we are using, we need to find an exact eigenfunction generator $\Psi(\nu)$ satisfying

$$Y(\Psi(\nu)) \equiv 1 \bmod \Delta(\nu).$$

Thanks to the explicit form of the vector field Y we can easily establish, arguing as in the proof of Lemma 5.2, the following equalities:

$$\begin{aligned} &J_i' := \mathrm{Lie}_Y(J_i) = J_i(c_1, 0, \ldots, 0, -c_n), \\ &J_i'' := \mathrm{Lie}_Y(J_i') = J_i(c_1, 0, \ldots, 0, c_n), \\ &\text{and, finally, } J_i''' = J_i'. \end{aligned} \tag{5.20}$$

This shows that the functions

$$\eta_i = \log(\nu_i), \quad \psi_i = \log\left(Y(\sum_k \nu_i^k \mathcal{H}_k) + Y^2(\sum_k \nu_i^k \mathcal{H}_k) \right), \; i = 1, \ldots, k.$$

provide a set of canonical DN coordinates for the Volterra lattice with $2k+1$ sites. An analogous result holds for the VL with an even number of sites, although the vector field Y has a more complicated expression.

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